

GLOBAL WELL-POSEDNESS FOR THE 2D MUSKAT PROBLEM WITH SLOPE LESS THAN 1.

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ABSTRACT. We prove the existence of global, smooth solutions to the 2D Muskat problem in the stable regime whenever the initial data has slope strictly less than 1. The curvature of these solutions decays to 0 as t goes to infinity, and they are unique when the initial data is $C^{1,\epsilon}$. We do this by getting a priori estimates using a nonlinear maximum principle first introduced in [10], where the authors proved global well-posedness for the quasi-geostrophic equation.

1. INTRODUCTION

The Muskat problem was originally introduced by Muskat in [11] in order to model the interface between water and oil in tar sands. In general, it describes the interface between two incompressible, immiscible fluids of different constant densities in a porous media. The fluids evolve according to Darcy's law, giving an evolution of the interface (see [5] for derivation of equations), and in 2D is analogous to the two phase Hele-Shaw cell (see [13]). In the case that the two fluids are of equal viscosity and the interface is given by the graph $y = f(t, x)$ with the denser fluid on bottom (i.e. the stable regime), the function f satisfies

$$(1.1) \quad f_t(t, x) = \int_{\mathbb{R}} \frac{(f_x(t, y) - f_x(t, x))(y - x)}{(f(t, y) - f(t, x))^2 + (y - x)^2} dy,$$

after the appropriate renormalization. By making a change of variables, (see the proof of Lemma 5.1 of [6]) we get the equivalent system

$$(1.2) \quad f_t(t, x) = \int_{\mathbb{R}} \frac{f(t, y) - f(t, x) - (y - x)f_x(t, x)}{(f(t, y) - f(t, x))^2 + (y - x)^2} dy,$$

which will be more useful for our purposes. Since the function f is Lipschitz, the above integral can be viewed as a nonlinear perturbation of the half Laplacian. In fact, it is easy to see that linearizing around a flat solution gives

$$(1.3) \quad f_t(t, x) = -c(-\Delta)^{1/2} f(t, x),$$

demonstrating the natural parabolicity of the problem.

The Muskat problem is known to be locally well-posed in H^k for $k \geq 3$ with solutions satisfying L^∞ and L^2 maximum principles, but neither imply any gain of derivatives (see [6], [3]).

Because solutions to (1.2) have the natural scaling $\frac{1}{r}f(rt, rx)$, we see that the slope $\|f_x\|_{L^\infty}$ is the critical quantity. Specifically, $\|f_x\|_{L^\infty} < 1$ appears to be a cutoff for many of the regularization effects, as we shall see in section 3 the derivative f_x obeys the equation

$$(1.4) \quad (f_x)_t(t, x) = f_{xx}(t, x) \int_{\mathbb{R}} \frac{-h}{\delta_h f(t, x)^2 + h^2} dh + \int_{\mathbb{R}} \delta_h f_x(t, x) K(t, x, h) dh.$$

where $\delta_h f(t, x) := f(t, x + h) - f(t, x)$ and the kernel K is uniformly elliptic of order 1 precisely when $\|f_x\|_{L^\infty} < 1$. Thus this is a natural cutoff point for the equation.

Under the assumption $\|f'_0\|_{L^\infty} < 1$, there have been a number of positive results. In [3] the authors prove an L^∞ maximal principle for the slope f_x along with the existence of global weak Lipschitz solutions. Recently, [8] improved the L^2 energy estimate of [3] (which holds for any solution) to one analogous with the energy estimate from the linear equation under this assumption on the slope. When the initial data $f_0 \in H^2(\mathbb{R})$ with $\|f_0\|_1 = \|\xi|\hat{f}_0(\xi)|\|_{L^1_\xi}$ less than some explicit constant $\approx 1/3$ (which implies slope less than 1), [2] proves that a unique global strong solution exists. In this case [12] proves optimal decay estimates on the norms $\|f(t, \cdot)\|_s = \|\xi|^s \hat{f}(t, \xi)\|_{L^1_\xi}$, matching the estimates for the linear equation.

In contrast to the positive results, [1] shows that there is an open subset of initial data in H^4 such that the Rayleigh-Taylor condition breaks down in finite time. That is, $\lim_{t \rightarrow t_0+} \|f_x(t, \cdot)\|_{L^\infty} = \infty$ for some time t_0 , after which the interface between the fluids can no longer be described by a graph.

The authors of [4] made great progress towards proving global regularity. They proved that if the initial data $f_0 \in H^k$, then the solution f will exist and remain in H^k so long as the slope $f_x(t, \cdot)$ remains bounded and uniformly continuous. Thus the natural next step is to prove the generation of a modulus of continuity for f_x , hence

Theorem 1.1. *Let $f_0 \in W^{1,\infty}(\mathbb{R})$ with $\|f'_0\|_{L^\infty} < 1$. Then there exists a classical solution*

$$(1.5) \quad f \in C([0, \infty) \times \mathbb{R}) \cap C^{1,\alpha}_{loc}((0, \infty) \times \mathbb{R}) \cap L^\infty_{loc}((0, \infty); C^{1,1}),$$

to (1.2) with f_x satisfying

$$(1.6) \quad f_x(t, x) - f_x(t, y) \leq \rho\left(\frac{|x - y|}{t}\right), \quad t > 0, x \neq y \in \mathbb{R},$$

for some Lipschitz modulus of continuity ρ depending solely on $\|f'_0\|_{L^\infty}$. In the case that $f'_0 \in C^{1,\epsilon}(\mathbb{R})$ for some $\epsilon > 0$, then the solution f is unique with $f \in L^\infty([0, \infty); C^{1,\epsilon})$.

The uniqueness statement follows essentially from the uniqueness theorem of [4]. We note in the appendix the few small changes needed to their proof in order to apply it here.

The most vital part of Theorem 1.1 is the spontaneous generation of the modulus $\rho(\cdot/t)$, as everything else will follow from that. The spontaneous generation/propagation of a general modulus of continuity has old roots as classical Holder estimates, but its only recently that the idea to tailor made moduli for specific equations emerged. The technique first appeared in [10], where the authors used it to prove global well-posedness for the surface quasi-geostrophic equation. It has had great success at proving regularity for a number of active scalar equations, that is equations of the form

$$(1.7) \quad \theta_t + (u \cdot \nabla)\theta + \mathcal{L}\theta = 0,$$

where u is a flow depending on θ and \mathcal{L} is some diffusive operator. See [9], [7] for a good overview of results using this method.

To date, these tailor made moduli have only been applied to cases where all the nonlinearity has been in the flow velocity u , and the diffusive term \mathcal{L} has been rather nice (typically $(-\Delta)^\alpha$, or at least a Fourier multiplier). We will be applying this method to f_x , which solves the active scalar equation (1.4). Note that in this equation, the kernel K defined in (3.4) is a highly nonlinear function of f, f_x . Thus this is the first time the method has been applied in a fully nonlinear equation.

We prove Theorem 1.1 by deriving a priori estimates for smooth solutions to (1.2) with initial data $f_0 \in C^\infty_c(\mathbb{R})$ depending primarily on $\|f'_0\|_{L^\infty}$. We prove enough estimates that by approximating in $W^{1,\infty}_{loc}$ with smooth compactly supported initial data, we get solutions f^ϵ which will converge along subsequences in C^1_{loc} to a solution f solving (1.2) for arbitrary initial data $f_0 \in W^{1,\infty}(\mathbb{R})$ with $\|f'_0\|_{L^\infty} < 1$.

The rest of the paper is organized as follows. We begin by repeating the breakthrough argument of [10] in Section 2. In Section 3, we differentiate (1.2) to derive the equation for f_x . In Section 4, we state how a modulus of continuity ω interacts with the equation in our main technical lemma. In Sections 5 and 6 we then derive the bounds on the drift and diffusion terms necessary to prove that lemma. In Section 7, we apply our main technical lemma to a specific modulus of continuity, and finally in Section 8 we complete the proof of (1.6) by choosing the correct modulus ρ . In Section 9, we then use (1.6) to prove a few estimates on regularity in time, guaranteeing enough compactness to prove that there are classical solutions for rough initial data. Finally in the appendix, we give a quick outline for how to modify the uniqueness proof of [4] to work for initial data $f_0 \in C^{1,\epsilon}(\mathbb{R})$ with $\|f'_0\|_{L^\infty} < 1$.

2. BREAKTHROUGH SCENARIO

Assume that $f_0 \in C_c^\infty(\mathbb{R})$ with $\|f'_0\|_{L^\infty} < 1$, so that there exists a solution $f \in C^1((0, T_+); H^k)$ for k arbitrarily large and some $T_+ > 0$ by [6]. Note that $\|f_x\|_{L^\infty} < 1$ by the maximum principle proven in [3]. Fix a Lipschitz modulus ρ which we will define later. For sufficiently small times, $f_x(t, \cdot)$ will have modulus $\rho(\cdot/t)$ since it is smooth and bounded. It then follows by the main theorem of [4] that as long as $f_x(t, \cdot)$ continues to have modulus $\rho(\cdot/t)$, the solution f will exist with $T_+ > t$.

So, we proceed as in [10]'s proof for quasi-geostrophic equation. Suppose that $f_x(t, \cdot)$ satisfies (1.6) for all $t < T$. Then by continuity,

$$(2.1) \quad f_x(T, x) - f_x(T, y) \leq \rho\left(\frac{|x - y|}{T}\right), \quad \forall x \neq y \in \mathbb{R}.$$

We first prove that if we have the strict inequality $f_x(T, x) - f_x(T, y) < \rho(|x - y|/T)$, then $f_x(t, \cdot)$ will have modulus $\rho(\cdot/t)$ for $t \leq T + \epsilon$.

Lemma 2.1. *Let $f \in C([0, T_+); C_0^3(\mathbb{R}))$, and $T \in (0, T_+)$. Suppose that $f(T, \cdot)$ satisfies*

$$(2.2) \quad f_x(T, x) - f_x(T, y) < \rho(|x - y|/T), \quad \forall x \neq y \in \mathbb{R},$$

for some Lipschitz modulus of continuity ρ with $\rho''(0) = -\infty$. Then

$$(2.3) \quad f_x(T + \epsilon, x) - f_x(T + \epsilon, y) < \rho(|x - y|/T + \epsilon), \quad \forall x \neq y \in \mathbb{R},$$

for all $\epsilon > 0$ sufficiently small.

Proof. To begin, note that for any compact subset $K \subset \mathbb{R}^2 \setminus \{(x, x) | x \in \mathbb{R}\}$,

$$(2.4) \quad f_x(T, x) - f_x(T, y) < \rho(|x - y|/T) \quad \forall (x, y) \in K \quad \Rightarrow \quad f_x(T + \epsilon, x) - f_x(T + \epsilon, y) < \rho(|x - y|/T + \epsilon) \quad \forall (x, y) \in K,$$

for $\epsilon > 0$ sufficiently small by uniform continuity. So, we only need to focus on pairs (x, y) that are either close to the diagonal, or that are large.

To handle (x, y) near the diagonal, we start by noting that since $f(T, \cdot) \in C^3(\mathbb{R})$ and $\rho''(0) = -\infty$, for every x we get that

$$(2.5) \quad |f_{xx}(T, x)| < \frac{\rho'(0)}{T}.$$

Since $f \in C([0, T_+); C_0^3(\mathbb{R}))$, $f_{xx}(T, x) \rightarrow 0$ as $x \rightarrow \infty$. Thus we can take the point where $\max_x |f_{xx}(T, x)|$ is achieved to get that

$$(2.6) \quad \|f_{xx}(T, \cdot)\|_{L^\infty} < \frac{\rho'(0)}{T}.$$

By continuity of f_{xx} , we thus have $\|f_{xx}(T + \epsilon, \cdot)\|_{L^\infty} < \frac{\rho'(0)}{T + \epsilon}$ for $\epsilon > 0$ sufficiently small. Hence,

$$(2.7) \quad f_x(T + \epsilon, x) - f_x(T + \epsilon, y) < \rho \left(\frac{|x - y|}{T + \epsilon} \right), \quad |x - y| < \delta,$$

for ϵ, δ sufficiently small.

Now let $R_1, R_2 > 0$ be such that

$$(2.8) \quad \rho(R_1/T + \epsilon) > \text{osc}_{\mathbb{R}} f_x(T + \epsilon, \cdot),$$

and that $|x| > R_2$ implies

$$(2.9) \quad |f_x(T + \epsilon, x)| < \frac{\rho(\delta/T + \epsilon)}{2},$$

for $\epsilon > 0$ sufficiently small. Taking $R = R_1 + R_2$, it's easy to check that $|x| > R$ implies that

$$(2.10) \quad |f_x(T + \epsilon, x) - f_x(T + \epsilon, y)| < \rho(|x - y|/T + \epsilon), \quad \forall y \neq x.$$

Finally, taking $K = \{(x, y) \in \mathbb{R}^2 : |x - y| \geq \delta, x, y \in \overline{B_R}\}$, we're done. \square

Thus by the lemma, if f_x was to lose its modulus after time T , we must have that there exist $x \neq y \in \mathbb{R}$ with

$$(2.11) \quad f_x(T, x) - f_x(T, y) = \rho \left(\frac{|x - y|}{T} \right).$$

We will show for a smooth solution f of (1.2) and the correct choice of ρ that in this case

$$(2.12) \quad \left. \frac{d}{dt} (f_x(t, x) - f_x(t, y)) \right|_{t=T} < \left. \frac{d}{dt} \left(\rho \left(\frac{|x - y|}{t} \right) \right) \right|_{t=T},$$

contradicting the fact that f_x had modulus $\rho(\cdot/t)$ for time $t < T$.

Thus we just need to prove (2.12) to complete the proof of the generation of modulus of continuity (1.6) of Theorem 1.1.

3. EQUATION FOR f_x

So, we just need to prove (2.12). To begin, we need to examine the equation that f_x solves. Since everything we will be doing is for some fixed time $T > 0$, we will suppress the time variable from now on. Differentiating (1.2), we see that f_x solves

$$(3.1) \quad \begin{aligned} (f_x)_t(x) &= f_{xx}(x) \int_{\mathbb{R}} \frac{x - y}{(f(y) - f(x))^2 + (y - x)^2} dy \\ &\quad + \int_{\mathbb{R}} (f(y) - f(x) - (y - x)f_x(x)) \frac{2((f(y) - f(x))f_x(x) + (y - x))}{((f(y) - f(x))^2 + (y - x)^2)^2} dy. \end{aligned}$$

To simplify notation, we reparametrize (3.1) by taking $y = x + h$, and letting

$$\delta_h f(x) := f(x + h) - f(x),$$

we get

$$(3.2) \quad \begin{aligned} (f_x)_t(x) &= f_{xx}(x) \int_{\mathbb{R}} \frac{-h}{(\delta_h f(x))^2 + h^2} dh \\ &\quad + \int_{\mathbb{R}} (\delta_h f(x) - hf_x(x)) \frac{2((\delta_h f(x))f_x(x) + h)}{(\delta_h f(x)^2 + h^2)^2} dh. \end{aligned}$$

Note that

$$\delta_h f(x) - hf_x(x) = \int_0^h \delta_s f_x(x) ds,$$

for $h > 0$, and

$$\delta_h f(x) - hf_x(x) = - \int_h^0 \delta_s f_x(x) ds,$$

for $h < 0$.

With that in mind, define

$$(3.3) \quad k(x, s) = \frac{2(\delta_s f(x)f_x(x) + s)}{(\delta_s f(x)^2 + s^2)^2},$$

and

$$(3.4) \quad K(x, h) = \begin{cases} \int_h^\infty k(x, s) ds, & h > 0 \\ \int_{-\infty}^h -k(x, s) ds, & h < 0 \end{cases}.$$

Then since $|\delta_s f(x)| \leq |s| \|f_x\|_{L^\infty}$, we see that

$$\frac{2(1 - \|f_x\|_{L^\infty}^2)}{(1 + \|f_x\|_{L^\infty}^2)^2} \frac{1}{|s|^3} \leq \operatorname{sgn}(s)k(x, s) \leq \frac{2(1 + \|f_x\|_{L^\infty}^2)}{|s|^3}.$$

Since $\|f'_0\|_{L^\infty} < 1$, the maximum principle from [3] tells us that $\|f_x\|_{L^\infty} \leq \|f'_0\|_{L^\infty} < 1$, and hence that

$$(3.5) \quad 0 < \frac{\lambda}{h^2} \leq K(x, h) \leq \frac{\Lambda}{h^2},$$

where

$$(3.6) \quad \lambda = \frac{1 - \|f'_0\|_{L^\infty}^2}{(1 + \|f'_0\|_{L^\infty}^2)^2}, \quad \Lambda = 1 + \|f'_0\|^2.$$

Thus K is comparable to the kernel for $(-\Delta)^{1/2}$. Note that the main reason we require $\|f'_0\|_{L^\infty} < 1$ is to ensure this ellipticity of K .

Integrating (3.1) by parts, we then get that f_x solves

$$(3.7) \quad (f_x)_t(x) = f_{xx}(x) \int_{\mathbb{R}} \frac{-h}{\delta_h f(x)^2 + h^2} dh + \int_{\mathbb{R}} \delta_h f_x(x) K(x, h) dh.$$

4. MODULI ESTIMATES

Our goal is to show that if $f_x(T, \cdot)$ has modulus $\rho(\cdot/T)$ and equality is achieved at two points (2.11), then (2.12) must hold, contradicting the assumptions of the breakthrough argument (see section 2). To that end, we first need to understand how a modulus of continuity interacts with the equation for f_x (3.7). Hence,

Lemma 4.1. *Let $f : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be a bounded smooth solution to (1.2) with $\|f'_0\|_{L^\infty} < 1$, and $\omega : [0, \infty) \rightarrow [0, \infty)$ be some fixed modulus of continuity. Assume that at some fixed time T that*

$$(4.1) \quad \begin{aligned} \delta_h f_x(T, x) &\leq \omega(|h|), \\ f_x(T, \xi/2) - f_x(T, -\xi/2) &= \omega(\xi), \end{aligned}$$

for all $h \in \mathbb{R}$, and for some $\xi > 0$. Then

(4.2)

$$\begin{aligned} \frac{d}{dt}(f_x(T, \xi/2) - f_x(T, -\xi/2)) \leq & A\omega'(\xi) \left(\int_0^\xi \frac{\omega(h)}{h} dh + \xi \int_\xi^\infty \frac{\omega(h)}{h^2} dh + \ln(M+1)\omega(\xi) \right) \\ & + A\omega(\xi) \int_{M\xi}^\infty \frac{\omega(h)}{h^2} dh + 2(\Lambda - \lambda) \int_\xi^{M\xi} \frac{(\omega(h-\xi) - \omega(\xi))_+}{h^2} dh \\ & + 2\lambda \int_0^\xi \frac{\delta_h \omega(\xi) + \delta_{-h} \omega(\xi)}{h^2} dh + 2\lambda \int_\xi^\infty \frac{\omega(h+\xi) - \omega(h) - \omega(\xi)}{h^2} dh, \end{aligned}$$

for any $M \geq 1$, where A depends only on $\|f'_0\|_{L^\infty}$ and λ, Λ are as in (3.6).

This is the main technical lemma that we need. Since solutions to (1.2) are closed under translation and sign change, it suffices to consider the above situation for our proof of (2.12).

The proof for Lemma 4.1 is essentially a nondivergence form argument; our function f_x is touched from above at $\xi/2$ by our modulus ω , and its touched from below at $-\xi/2$ by $-\omega$. Specifically,

$$(4.3) \quad \begin{aligned} \delta_h f_x(\xi/2) &\leq \delta_h \omega(\xi), & \forall h > -\xi, \\ \delta_h f_x(-\xi/2) &\geq -\delta_{-h} \omega(\xi), & \forall h < \xi. \end{aligned}$$

From (4.3), we want to derive as much information as we can and bound $\frac{d}{dt}(f_x(\xi/2) - f_x(-\xi/2))$. To that end, by dividing (4.3) through by h and taking the limit as $h \rightarrow 0$, we then get that

$$(4.4) \quad f_{xx}(\xi/2) = f_{xx}(-\xi/2) = \omega'(\xi).$$

Hence by our equation for f_x (3.7), we have that

(4.5)

$$\begin{aligned} \frac{d}{dt}(f_x(\xi/2) - f_x(-\xi/2)) &= \omega'(\xi) \int_{\mathbb{R}} \left(\frac{-h}{\delta_h f(\xi/2)^2 + h^2} - \frac{-h}{\delta_h f(-\xi/2)^2 + h^2} \right) dh \\ &\quad + \int_{\mathbb{R}} \delta_h f_x(\xi/2) K(\xi/2, h) - \delta_h f_x(-\xi/2) K(-\xi/2, h) dh \\ &= \omega'(\xi) \int_{\mathbb{R}} \left(\frac{-h}{\delta_h f(\xi/2)^2 + h^2} - \frac{-h}{\delta_h f(-\xi/2)^2 + h^2} \right) dh + \omega'(\xi) \int_{-M\xi}^{M\xi} (hK(\xi/2, h) - hK(-\xi/2, h)) dh \\ &\quad + \int_{-M\xi}^{M\xi} (\delta_h f_x(\xi/2) - h\omega'(\xi)) K(\xi/2, h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi)) K(-\xi/2, h) dh \\ &\quad + \int_{|h| > M\xi} \delta_h f_x(\xi/2) K(\xi/2, h) - \delta_h f_x(-\xi/2) K(-\xi/2, h) dh, \end{aligned}$$

for any $M \geq 1$. The first two terms of the RHS of (4.5) act as a drift, giving rise to the first two error terms of (4.2). The latter two terms of (4.5) act as a diffusion, giving rise to both the helpful (negative) terms in (4.2), as well as additional error terms (the middle terms of (4.2)) arising from the difference in the kernels, $|K(\xi/2, h) - K(-\xi/2, h)|$.

5. BOUNDS ON DRIFT TERMS

We begin proving Lemma 4.1 by bounding the drift terms of (4.5), starting with

Lemma 5.1. *Under the assumptions of Lemma 4.1,*

$$(5.1) \quad \omega'(\xi) \left| \int_{\mathbb{R}} \frac{-h}{\delta_h f(\xi/2)^2 + h^2} - \frac{-h}{\delta_h f(-\xi/2)^2 + h^2} dh \right| \lesssim \omega'(\xi) \left(\int_0^\xi \frac{\omega(h)}{h} dh + \xi \int_\xi^\infty \frac{\omega(h)}{h^2} dh \right).$$

Proof. We want to bound (5.1) by symmetrizing the kernels for $|h| < \xi$, and then using the continuity in the first variable for $|h| > \xi$. To that end,

$$(5.2) \quad \begin{aligned} \omega'(\xi) \int_{\mathbb{R}} \left(\frac{-h}{\delta_h f(\xi/2)^2 + h^2} - \frac{-h}{\delta_h f(-\xi/2)^2 + h^2} \right) dh \\ \leq \omega'(\xi) \int_0^\xi h \left| \frac{\delta_h f(\xi/2)^2 - \delta_{-h} f(\xi/2)^2}{(\delta_h f(\xi/2)^2 + h^2)(\delta_{-h} f(\xi/2)^2 + h^2)} + \frac{\delta_h f(-\xi/2)^2 - \delta_{-h} f(-\xi/2)^2}{(\delta_h f(-\xi/2)^2 + h^2)(\delta_{-h} f(-\xi/2)^2 + h^2)} \right| dh \\ + \omega'(\xi) \int_{|h|>\xi} |h| \left| \frac{\delta_h f(\xi/2)^2 - \delta_h f(-\xi/2)^2}{(\delta_h f(\xi/2)^2 + h^2)(\delta_h f(-\xi/2)^2 + h^2)} \right| dh. \end{aligned}$$

We bound the first integral using

$$|\delta_h f(x)| \lesssim |h|,$$

$$(5.3) \quad |\delta_h f(x) + \delta_{-h} f(x)| = \left| \int_0^h f_x(x+s) - f_x(x+s-h) ds \right| \leq \omega(h)h,$$

Thus get that for $0 \leq h < \xi$,

$$(5.4) \quad \left| \frac{\delta_h f(x)^2 - \delta_{-h} f(x)^2}{(\delta_h f(x)^2 + h^2)(\delta_{-h} f(x)^2 + h^2)} \right| \lesssim \frac{\omega(h)}{h^2},$$

and hence

$$(5.5) \quad \int_0^\xi h \left| \frac{\delta_h f(\xi/2)^2 - \delta_{-h} f(\xi/2)^2}{(\delta_h f(\xi/2)^2 + h^2)(\delta_{-h} f(\xi/2)^2 + h^2)} \right| dh \lesssim \int_0^\xi \frac{\omega(h)}{h} dh.$$

For $|h| \geq \xi$, we bound $|\delta_h f(\xi/2) + \delta_h f(-\xi/2)| \lesssim |h|$ and

$$(5.6) \quad \begin{aligned} \left| \delta_h f(\xi/2) - \delta_h f(-\xi/2) \right| &= \left| \int_0^h f_x(\xi/2+s) - f_x(-\xi/2+s) ds \right| \\ &= \left| \int_0^\xi f_x(h-\xi/2+s) - f_x(-\xi/2+s) ds \right| \leq \xi \omega(|h|), \end{aligned}$$

in order to get

$$(5.7) \quad \int_{|h|>\xi} |h| \left| \frac{\delta_h f(\xi/2)^2 - \delta_h f(-\xi/2)^2}{(\delta_h f(\xi/2)^2 + h^2)(\delta_h f(-\xi/2)^2 + h^2)} \right| dh \lesssim \xi \int_\xi^\infty \frac{\omega(h)}{h^2} dh.$$

Putting (5.5) and (5.7) together, we thus have

$$(5.8) \quad \omega'(\xi) \int_{\mathbb{R}} \left(\frac{-h}{\delta_h f(\xi/2)^2 + h^2} - \frac{-h}{\delta_h f(-\xi/2)^2 + h^2} \right) dh \lesssim \omega'(\xi) \left(\int_0^\xi \frac{\omega(h)}{h} dh + \xi \int_\xi^\infty \frac{\omega(h)}{h^2} dh \right).$$

□

That leaves us with the second drift term of (4.5),

Lemma 5.2. *Under the assumptions of Lemma 4.1, for any $M \geq 1$*

$$(5.9) \quad \omega'(\xi) \left| \int_{-M\xi}^{M\xi} hK(\xi/2, h) - hK(-\xi/2, h) dh \right| \lesssim \omega'(\xi) \left(\int_0^\xi \frac{\omega(h)}{h} dh + \xi \int_\xi^\infty \frac{\omega(h)}{h^2} dh + \ln(M+1)\omega(\xi) \right).$$

Proof. To begin, we note

$$(5.10) \quad \omega'(\xi) \left| \int_{-M\xi}^{M\xi} hK(\xi/2, h) - hK(-\xi/2, h) dh \right| \leq \omega'(\xi) \int_0^{M\xi} h \left| K(\xi/2, h) - K(\xi/2, -h) - K(-\xi/2, h) + K(-\xi/2, -h) \right| dh.$$

Recall the definition of K , (3.4),

$$(5.11) \quad K(x, h) = \begin{cases} \int_h^\infty k(x, s) ds, & h > 0 \\ \int_{-\infty}^h -k(x, s) ds, & h < 0 \end{cases},$$

$$k(x, s) = \frac{2(\delta_s f(x) f_x(x) + s)}{(\delta_s f(x)^2 + s^2)^2}.$$

So, to control (5.10) we first need to bound $|k(x, s) + k(x, -s)|$ for $0 \leq s < \xi$, and $|k(\xi/2, s) - k(-\xi/2, s)|$ for $|s| > \xi$. For the first, using the bounds (5.3) we see that

$$(5.12) \quad \begin{aligned} |k(x, s) + k(x, -s)| &= \left| \frac{2(\delta_s f(x) f_x(x) + s)}{(\delta_s f(x)^2 + s^2)^2} + \frac{2(\delta_{-s} f(x) f_x(x) - s)}{(\delta_{-s} f(x)^2 + s^2)^2} \right| \\ &\leq \frac{2|\delta_s f(x) + \delta_{-s} f(x)| \cdot |f_x(x)|}{(\delta_{-s} f(x)^2 + s^2)^2} + 2|\delta_s f(x) f_x(x) + s| \left| \frac{(\delta_s f(x)^2 + s^2)^2 - (\delta_{-s} f(x)^2 + s^2)^2}{(\delta_s f(x)^2 + s^2)^2 (\delta_{-s} f(x)^2 + s^2)^2} \right| \\ &\lesssim \frac{\omega(s)}{s^3} + s \left| \frac{\delta_s f(x)^4 - \delta_{-s} f(x)^4 + 2s^2(\delta_s f(x)^2 - \delta_{-s} f(x)^2)}{s^8} \right| \\ &\lesssim \frac{\omega(s)}{s^3}. \end{aligned}$$

For the second, using (5.3), (5.6), and (4.1) we get that

$$\begin{aligned}
(5.13) \quad |k(\xi/2, s) - k(-\xi/2, s)| &= \left| \frac{2(\delta_s f(\xi/2)f_x(\xi/2) + s)}{(\delta_s f(\xi/2)^2 + s^2)^2} - \frac{2(\delta_s f(-\xi/2)f_x(-\xi/2) + s)}{(\delta_s f(-\xi/2)^2 + s^2)^2} \right| \\
&\leq 2 \frac{|\delta_s f(\xi/2)f_x(\xi/2) - \delta_s f(-\xi/2)f_x(-\xi/2)|}{(\delta_s f(-\xi/2)^2 + s^2)^2} \\
&\quad + 2|\delta_s f(\xi/2)f_x(\xi/2) + s| \left| \frac{(\delta_s f(\xi/2)^2 + s^2)^2 - (\delta_s f(-\xi/2)^2 + s^2)^2}{(\delta_s f(\xi/2)^2 + s^2)^2 (\delta_s f(-\xi/2)^2 + s^2)^2} \right| \\
&\lesssim \frac{|\delta_s f(\xi/2) - \delta_s f(-\xi/2)| \cdot |f_x(\xi/2)|}{s^4} + \frac{|\delta_s f(-\xi/2)| \cdot |f_x(\xi/2) - f_x(-\xi/2)|}{s^4} \\
&\quad + |s| \left| \frac{\delta_s f(\xi/2)^4 - \delta_s f(-\xi/2)^4 + s^2(\delta_s f(\xi/2)^2 - \delta_s f(-\xi/2)^2)}{s^8} \right| \\
&\lesssim \frac{\xi \omega(s)}{s^4} + \frac{\omega(\xi)}{s^3}.
\end{aligned}$$

So using (5.12) and (5.13), we can first bound

$$\begin{aligned}
(5.14) \quad \int_0^\xi h \left| K(\xi/2, h) - K(\xi/2, -h) - K(-\xi/2, h) + K(-\xi/2, -h) \right| dh \\
\lesssim \int_0^\xi h \int_h^\xi \frac{\omega(s)}{s^3} ds dh + \int_0^\xi h \int_\xi^\infty \frac{\xi \omega(s)}{s^4} + \frac{\omega(\xi)}{s^3} ds dh \\
\lesssim \int_0^\xi \frac{\omega(s)}{s^3} \int_0^s h dh ds + \int_\xi^\infty \frac{\xi^3 \omega(s)}{s^4} + \frac{\xi^2 \omega(\xi)}{s^3} ds \\
\lesssim \int_0^\xi \frac{\omega(s)}{s} ds + \xi \int_\xi^\infty \frac{\omega(s)}{s^2} ds + \omega(\xi).
\end{aligned}$$

For the rest of (5.10), we use (5.13) again to also bound

$$\begin{aligned}
(5.15) \quad \left| \int_{M\xi > |h| > \xi} h(K(\xi/2, h) - K(-\xi/2, h)) dh \right| &\lesssim \int_\xi^{M\xi} h \int_h^\infty \frac{\omega(\xi)}{s^3} + \frac{\xi \omega(s)}{s^4} ds \\
&\lesssim \omega(\xi) \int_\xi^{M\xi} \frac{1}{h} dh + \xi \int_\xi^{M\xi} \frac{\omega(h)}{h^2} dh \\
&\lesssim \ln(M) \omega(\xi) + \xi \int_\xi^\infty \frac{\omega(h)}{h^2} dh.
\end{aligned}$$

□

6. BOUNDS ON DIFFUSIVE TERMS

Now we move on to proving an upper bound for the diffusive terms of (4.5). We can rewrite them as

$$\begin{aligned}
(6.1) \quad & \int_{-M\xi}^{M\xi} (\delta_h f_x(\xi/2) - h\omega'(\xi))K(\xi/2, h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi))K(-\xi/2, h)dh \\
& + \int_{|h|>M\xi} \delta_h f_x(\xi/2)K(\xi/2, h) - \delta_h f_x(-\xi/2)K(-\xi/2, h)dh \\
& = \int_{-M\xi}^{M\xi} (\delta_h f_x(\xi/2) - h\omega'(\xi))K(\xi/2, h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi))K(-\xi/2, h)dh \\
& + \int_{|h|>M\xi} [\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)]K(\xi/2, h)dh + \int_{|h|>M\xi} \delta_h f_x(-\xi/2)[K(\xi/2, h) - K(-\xi/2, h)]dh.
\end{aligned}$$

We begin by bounding the last term, which is an error term.

Lemma 6.1. *Under the assumptions of Lemma 4.1,*

$$(6.2) \quad \left| \int_{|h|>M\xi} \delta_h f_x(-\xi/2)[K(\xi/2, h) - K(-\xi/2, h)]dh \right| \lesssim \omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2}dh + \omega'(\xi)\xi \int_{\xi}^{\infty} \frac{\omega(h)}{h^2}dh.$$

Proof. Using the fact that f_x has modulus ω and the bounds 5.13, it follows that

$$\begin{aligned}
(6.3) \quad & \int_{|h|>M\xi} \delta_h f_x(-\xi/2)[K(\xi/2, h) - K(-\xi/2, h)]dh \lesssim \int_{M\xi}^{\infty} \omega(h) \int_h^{\infty} \frac{\omega(\xi)}{s^3} + \frac{\xi\omega(s)}{s^4} ds dh \\
& \lesssim \omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2}dh + \int_{M\xi}^{\infty} \omega(h) \int_h^{\infty} \frac{\xi\omega(\xi) + \xi\omega'(\xi)(s-\xi)}{s^4} ds dh \\
& \lesssim \omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2}dh + \omega(\xi) \int_{M\xi}^{\infty} \frac{\xi\omega(h)}{h^3}dh + \omega'(\xi)\xi \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2}dh \\
& \lesssim \omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2}dh + \omega'(\xi)\xi \int_{\xi}^{\infty} \frac{\omega(h)}{h^2}dh.
\end{aligned}$$

□

For the other two terms in (6.1), we bound them in two stages.

Lemma 6.2. *Under the assumptions of Lemma 4.1,*

$$\begin{aligned}
(6.4) \quad & \int_{-M\xi}^{M\xi} (\delta_h f_x(\xi/2) - h\omega'(\xi))K(\xi/2, h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi))K(-\xi/2, h)dh \\
& + \int_{|h|>M\xi} [\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)] K(\xi/2, h)dh \\
& \leq \lambda \int_{\mathbb{R}} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh + 2(\Lambda - \lambda) \int_{\xi}^{M\xi} \frac{(\omega(h - \xi) - \omega(\xi))_+}{h^2} dh \\
& \quad + \omega'(\xi) \left| \int_{\xi < |h| < M\xi} h [K(\xi/2, h) - K(-\xi/2, h)] \right|.
\end{aligned}$$

Proof. We can bound the second term of (6.4) rather easily. Since

$$(6.5) \quad \delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2) = (f(h + \xi/2) - f(h - \xi/2)) - \omega(\xi) \leq 0,$$

by the uniform ellipticity of K ,

$$(6.6) \quad \int_{|h|>M\xi} [\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)] K(\xi/2, h)dh \leq \lambda \int_{|h|>M\xi} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh.$$

To bound the first term, we first define

$$(6.7) \quad G(\xi, h) = (\delta_h f_x(\xi/2) - h\omega'(\xi))K(\xi/2, h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi))K(-\xi/2, h).$$

Note that since ω touches f_x from above (see (4.3)), it follows that

$$(6.8) \quad \begin{aligned} \delta_h f_x(\xi/2) - \omega'(\xi)h &\leq \delta_h \omega(\xi) - \omega'(\xi)h \leq 0, & h \geq -\xi \\ \delta_h f_x(-\xi/2) - \omega'(\xi)h &\geq -\delta_{-h} \omega(\xi) - h\omega'(\xi) \geq 0, & h \leq \xi \end{aligned}$$

Thus for $|h| \leq \xi$, by the uniform ellipticity of K we have the bound

$$(6.9) \quad G(\xi, h) \leq \lambda \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2}.$$

That just leaves us with the case $\xi \leq |h| \leq M\xi$ to analyze. Note that we can write G in two distinct ways:

$$\begin{aligned}
(6.10) \quad G(\xi, h) &= (\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2))K(\xi/2, h) + (\delta_h f_x(-\xi/2) - h\omega'(\xi))(K(\xi/2, h) - K(-\xi/2, h)) \\
&= (\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2))K(-\xi/2, h) + (\delta_h f_x(\xi/2) - h\omega'(\xi))(K(\xi/2, h) - K(-\xi/2, h)).
\end{aligned}$$

By (6.8), $\delta_h f_x(\xi/2) - h\omega'(\xi) \leq 0$ for all $h > \xi$. Thus if $K(\xi/2, h) - K(-\xi/2, h) \geq 0$, then

$$(6.11) \quad G(\xi, h) \leq \lambda \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2}, \quad \text{if } K(\xi/2, h) - K(-\xi/2, h) \geq 0$$

On the otherhand, since

$$(6.12) \quad \delta_h f_x(-\xi/2) = \delta_{h-\xi} f(\xi/2) + \omega(\xi) \geq -\omega(h - \xi) + \omega(\xi)$$

for $h \geq \xi$, we see that

(6.13)

$$\begin{aligned} G(\xi, h) &\leq \lambda \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} + (\delta_h f_x(-\xi/2) - h\omega'(\xi))(K(\xi/2, h) - K(-\xi/2, h)) \\ &\leq \lambda \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} + (\Lambda - \lambda) \frac{(\omega(h - \xi) - \omega(\xi))_+}{h^2} + h\omega'(\xi)|K(\xi/2, h) - K(-\xi/2, h)|, \\ &\quad \text{if } K(\xi/2, h) - K(-\xi/2, h) \leq 0. \end{aligned}$$

Putting these two together, we get that

(6.14)

$$G(\xi, h) \leq \lambda \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} + (\Lambda - \lambda) \frac{(\omega(h - \xi) - \omega(\xi))_+}{h^2} + h\omega'(\xi)|K(\xi/2, h) - K(-\xi/2, h)|.$$

for $h \geq \xi$. A similar argument can be made in the case that $h \leq -\xi$.

Putting this all together,

$$\begin{aligned} (6.15) \quad &\int_{-M\xi}^{M\xi} G(\xi, h)dh + \int_{|h| > M\xi} [\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)] K(\xi/2, h)dh \\ &\leq \lambda \int_{\mathbb{R}} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh + 2(\Lambda - \lambda) \int_{\xi}^{M\xi} \frac{(\omega(h - \xi) - \omega(\xi))_+}{h^2} dh \\ &\quad + \omega'(\xi) \int_{\xi < |h| < M\xi} \left| h [K(\xi/2, h) - K(-\xi/2, h)] \right| dh. \end{aligned}$$

□

It's clear that we can bound $\int_{\xi < |h| < M\xi} \left| h [K(\xi/2, h) - K(-\xi/2, h)] \right| dh$ as in (5.15). Thus the only thing remaining to prove (4.2) is

Lemma 6.3. *Under the assumptions of Lemma 4.1,*

(6.16)

$$\lambda \int_{\mathbb{R}} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh \leq 2\lambda \int_0^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h} \omega(\xi)}{h^2} dh + 2\lambda \int_{\xi}^{\infty} \frac{\omega(\xi + h) - \omega(h) - \omega(\xi)}{h^2} dh.$$

Proof. To see this, note that formally we should have

$$(6.17) \quad \int_{\mathbb{R}} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh = \int_{\mathbb{R}} f_x(y) \left(\frac{1}{(y - \xi/2)^2} - \frac{1}{(y + \xi/2)^2} \right) - \frac{\omega(\xi)}{y^2} dy.$$

Thus in order to get an upper bound on the integral, we should be taking an upper bound on $f_x(y)$ when $y > 0$ and a lower bound when $y < 0$. Note by (4.3) that

$$(6.18) \quad \begin{aligned} f_x(y) &\leq f_x(\xi/2) + \omega(y + \xi/2) - \omega(\xi) = f_x(-\xi/2) + \omega(y + \xi/2), \quad y > -\xi/2, \\ f_x(y) &\geq f_x(-\xi/2) - \omega(-y - \xi/2) + \omega(\xi) = f_x(\xi/2) - \omega(-y - \xi/2), \quad y < \xi/2. \end{aligned}$$

In particular, this gives that

$$(6.19) \quad \int_{\xi}^{\infty} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh \leq \int_{\xi}^{\infty} \frac{\omega(\xi + h) - \omega(h) - \omega(\xi)}{h^2} dh,$$

and similarly for $\int_{-\infty}^{-\xi}$.

Combining this with the fact that for $|h| \leq \xi$

$$(6.20) \quad \delta_h f(\xi/2) - \delta_h f(-\xi/2) \leq \delta_h \omega(\xi) + \delta_{-h} \omega(\xi),$$

which is consistent with the upper/lower bounds we've used so far, we also get

$$(6.21) \quad \int_{-\xi}^{\xi} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh \leq \int_{-\xi}^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h} \omega(\xi)}{h^2} dh.$$

Thus

$$(6.22) \quad \lambda \int_{\mathbb{R}} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh \leq 2\lambda \int_0^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h} \omega(\xi)}{h^2} dh + 2\lambda \int_{\xi}^{\infty} \frac{\omega(\xi+h) - \omega(h) - \omega(\xi)}{h^2} dh.$$

□

7. MODULUS INEQUALITY

Combining all the estimates from the previous two sections, we get a proof of Lemma 4.1. Thus under the assumptions (4.1), we have that

$$(7.1) \quad \begin{aligned} \frac{d}{dt}(f_x(\xi/2) - f_x(-\xi/2)) &\leq A\omega'(\xi) \left(\int_0^{\xi} \frac{\omega(h)}{h} dh + \xi \int_{\xi}^{\infty} \frac{\omega(h)}{h^2} dh + \ln(M+1)\omega(\xi) \right) \\ &\quad + A\omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2} dh + 2(\Lambda - \lambda) \int_{\xi}^{M\xi} \frac{(\omega(h-\xi) - \omega(\xi))_+}{h^2} dh \\ &\quad + 2\lambda \int_0^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h} \omega(\xi)}{h^2} dh + 2\lambda \int_{\xi}^{\infty} \frac{\omega(h+\xi) - \omega(h) - \omega(\xi)}{h^2} dh, \end{aligned}$$

for any $M \geq 1$, where A is a constant depending only on $\|f'_0\|_{L^\infty}$.

In [10], the authors showed that the modulus

$$(7.2) \quad \begin{cases} \omega(\xi) = \xi - \xi^{3/2}, & 0 \leq \xi \leq \delta \\ \omega'(\xi) = \frac{\gamma}{\xi(4 + \log(\xi/\delta))}, & \xi \geq \delta \end{cases},$$

satisfies

$$(7.3) \quad A\omega'(\xi) \left(\int_0^{\xi} \frac{\omega(h)}{h} dh + \xi \int_{\xi}^{\infty} \frac{\omega(h)}{h^2} dh \right) + \lambda \int_0^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h} \omega(\xi)}{h^2} dh + \lambda \int_{\xi}^{\infty} \frac{\omega(h+\xi) - \omega(h) - \omega(\xi)}{h^2} dh < 0,$$

for all $\xi \in \mathbb{R}$ so long as δ, γ are sufficiently small.

With that in mind, we will show that

Lemma 7.1. *Under the assumptions of Lemma 4.1 for the modulus ω defined in (7.2),*

$$(7.4) \quad \frac{d}{dt}(f_x(\xi/2) - f_x(-\xi/2)) < -\omega'(\xi)\omega(\xi),$$

as long as δ, γ are taken sufficiently small depending on $\|f'_0\|_{L^\infty}$.

Proof. By the Lemma 4.1 and (7.3) which was proven in [10], it suffices to show

$$(7.5) \quad \begin{aligned} & A\omega'(\xi) \ln(M+1)\omega(\xi) + A\omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2} dh + 2(\Lambda - \lambda) \int_{\xi}^{M\xi} \frac{(\omega(h - \xi) - \omega(\xi))_+}{h^2} dh \\ & + \lambda \int_0^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h} \omega(\xi)}{h^2} dh + \lambda \int_{\xi}^{\infty} \frac{\omega(h + \xi) - \omega(h) - \omega(\xi)}{h^2} dh \leq -\omega'(\xi)\omega(\xi) \end{aligned}$$

for the correct choices of M , and δ, γ sufficiently small.

We proceed very similarly to [10]. To begin, for $\xi \leq \delta$ we take $M = 1$. Then we just need to show that

$$(7.6) \quad A\omega'(\xi)\omega(\xi) + A\omega(\xi) \int_{\xi}^{\infty} \frac{\omega(h)}{h^2} dh + \lambda \int_0^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h} \omega(\xi)}{h^2} dh + \lambda \int_{\xi}^{\infty} \frac{\omega(h + \xi) - \omega(h) - \omega(\xi)}{h^2} dh \leq -\omega'(\xi)\omega(\xi).$$

In this regime, note that we have the bounds

$$(7.7) \quad \left\{ \begin{array}{l} \int_{\xi}^{\delta} \frac{\omega(h)}{h^2} dh \leq \log(\delta/\xi), \\ \int_{\delta}^{\infty} \frac{\omega(h)}{h^2} dh = \frac{\omega(\delta)}{\delta} + \gamma \int_{\delta}^{\infty} \frac{1}{h^2(4 + \log(h/\delta))} dh \leq 1 + \frac{\gamma}{4\delta} \leq 2 \text{ if you take } \gamma < 4\delta, \\ \omega'(\xi) \leq 1, \\ \omega(\xi) \leq \xi, \\ \int_0^{\xi} \frac{\omega(\xi+h) + \omega(\xi-h) - 2\omega(\xi)}{h^2} dh \leq \xi \omega''(\xi) = -\frac{3}{2} \xi \xi^{-1/2}. \end{array} \right.$$

Putting this all together, we get that

$$(7.8) \quad \begin{aligned} & (A+1)\omega'(\xi)\omega(\xi) + A\omega(\xi) \int_{\xi}^{\infty} \frac{\omega(h)}{h^2} dh + \lambda \int_0^{\xi} \frac{\omega(\xi+h) + \omega(\xi-h) - 2\omega(\xi)}{h^2} dh \\ & + \lambda \int_{\xi}^{\infty} \frac{\omega(\xi+h) - \omega(h) - \omega(\xi)}{h^2} dh \leq \xi \left((A+1)(3 + \log(\delta/\xi)) - \frac{3}{2} \lambda \xi^{-1/2} \right) < 0, \end{aligned}$$

assuming that δ is sufficiently small.

Now assume that $\xi \geq \delta$. Then what we need to show is

$$(7.9) \quad \begin{aligned} & A\omega'(\xi) \ln(M+1)\omega(\xi) + A\omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2} dh + 2(\Lambda - \lambda) \int_{\xi}^{M\xi} \frac{(\omega(h - \xi) - \omega(\xi))_+}{h^2} dh \\ & + \lambda \int_0^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h} \omega(\xi)}{h^2} dh + \lambda \int_{\xi}^{\infty} \frac{\omega(h + \xi) - \omega(h) - \omega(\xi)}{h^2} dh \leq -\omega'(\xi)\omega(\xi). \end{aligned}$$

We first bound our new error terms. Using the definition of ω and integrating by parts, we see that

$$(7.10) \quad \begin{aligned} 2(\Lambda - \lambda) \int_{2\xi}^{M\xi} \frac{\omega(h - \xi) - \omega(\xi)}{h^2} dh &\leq 2(\Lambda - \lambda) \int_{\xi}^{\infty} \frac{\omega(h) - \omega(\xi)}{h^2} dh \\ &\leq 2(\Lambda - \lambda) \int_{\xi}^{\infty} \frac{\gamma}{h^2(4 + \log(h/\delta))} dh \leq \frac{2(\Lambda - \lambda)\gamma}{\xi} \leq \frac{\lambda}{4} \frac{\omega(\delta)}{\xi} \leq \frac{\lambda}{4} \frac{\omega(\xi)}{\xi}, \end{aligned}$$

assuming $\gamma \leq \frac{\lambda}{8(\Lambda - \lambda)}\omega(\delta)$.

In order to bound our other new error term, we will be taking M sufficiently large and then γ sufficiently small depending on M, δ . Noting that $\omega(\xi) \leq 2\|f_x\|_{L^\infty} < 2$, we can bound our other new error term by integrating by parts

$$(7.11) \quad \begin{aligned} A\omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2} dh &\leq \frac{2A}{M} \frac{\omega(M\xi)}{\xi} + 2A \int_{M\xi}^{\infty} \frac{\gamma}{h^2(4 + \log(h/\delta))} dh \\ &\leq \frac{2A}{M} \frac{\omega(M\xi)}{\xi} + \frac{2A}{M} \frac{\gamma}{\xi} \\ &\leq \frac{\lambda}{16} \frac{\omega(M\xi)}{\xi} + \frac{\lambda}{8} \frac{\omega(\xi)}{\xi}, \end{aligned}$$

assuming that $M \geq \frac{32A}{\lambda}$, and then γ is sufficiently small so that $\frac{2A}{M}\gamma \leq \frac{\lambda}{8}\omega(\delta) \leq \frac{\lambda}{8}\omega(\xi)$. Note that this is where we set a value for M , and that γ is taken sufficiently small depending on M . Now that the value for M is fixed, we can also control the value $\omega(M\xi)$ by taking γ sufficiently small that

$$(7.12) \quad \begin{aligned} \omega(M\xi) &= \omega(\xi) + \int_{\xi}^{M\xi} \frac{\gamma}{h(4 + \log(h/\delta))} dh \leq \omega(\xi) + \gamma \ln(M) \leq \omega(\xi) + \omega(\delta) \\ &\leq 2\omega(\xi). \end{aligned}$$

Hence,

$$(7.13) \quad A\omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2} dh \leq \frac{\lambda}{16} \frac{\omega(M\xi)}{\xi} + \frac{\lambda}{8} \frac{\omega(\xi)}{\xi} \leq \frac{\lambda}{4} \frac{\omega(\xi)}{\xi}.$$

Using the same integration by parts tricks, we can also show

$$(7.14) \quad \lambda \int_{\xi}^{\infty} \frac{\omega(h + \xi) - \omega(h) - \omega(\xi)}{h^2} dh \leq -\frac{3}{4} \lambda \frac{\omega(\xi)}{\xi}.$$

for γ sufficiently small.

So combining these together, we get that

$$(7.15) \quad A\omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2} dh + 2(\Lambda - \lambda) \int_{2\xi}^{M\xi} \frac{\omega(h - \xi) - \omega(\xi)}{h^2} dh + \lambda \int_{\xi}^{\infty} \frac{\omega(h + \xi) - \omega(h) - \omega(\xi)}{h^2} dh \leq \frac{-\lambda}{4} \frac{\omega(\xi)}{\xi}.$$

Since $\omega'(\xi)\omega(\xi) \leq \frac{\gamma\omega(\xi)}{\xi}$, we finally get that

$$(7.16) \quad (A \ln(M+1) + 1)\omega'(\xi)\omega(\xi) - \frac{\lambda}{4} \frac{\omega(\xi)}{\xi} \leq \frac{\omega(\xi)}{\xi} ((A \ln(M+1) + 1)\gamma - \lambda/4) < 0,$$

if γ is taken sufficiently small. \square

8. OUR CHOICE FOR THE MODULUS ρ

We've now shown that for the modulus defined in (7.2) that if the assumptions (4.1) hold that

$$(8.1) \quad \frac{d}{dt} (f_x(T, \xi/2) - f_x(T, -\xi/2)) < -\omega'(\xi)\omega(\xi).$$

First note that since $f(t, x)$ solves (1.2), then so does $\frac{f(rt, rx)}{r}$. Thus $f_x(rt, rx)$ will solve (3.7), and since all the constants in Lemma 4.1 only depended on $\|f_x\|_{L^\infty}$ which is preserved by the rescaling, we see that (8.1) will hold for any rescaling $\omega_r(h) = \omega(rh)$ as well. Also note that for $f_x(T, \xi/2) - f_x(T, -\xi/2) = \omega(\xi)$ to hold, we must necessarily have $\omega(\xi) \leq 2\|f_x(T, \cdot)\|_{L^\infty} < 2$. Thus taking

$$(8.2) \quad C = \sup_{0 < h < \omega^{-1}(2)} \frac{h}{\omega(h)} = \frac{\omega^{-1}(2)}{2},$$

we see that

$$(8.3) \quad \omega(h) \geq \frac{h}{C}.$$

for all relevant h . Define

$$(8.4) \quad \rho(h) := \omega(Ch),$$

so that

$$(8.5) \quad \rho(h) \geq h,$$

for all $h \in [0, \rho^{-1}(2)]$.

Now, suppose that at time T , f satisfies the assumptions (4.1) for $\rho(\cdot/T)$. Then since $\rho(\cdot/T)$ is a rescaling of ω , we have that

$$(8.6) \quad \frac{d}{dt} (f_x(T, \xi/2) - f_x(T, -\xi/2)) < -\frac{d}{dh} \rho(h/T) \Big|_{h=\xi} \rho(\xi/T) = \frac{-1}{T} \rho'(\xi/T) \rho(\xi/T) \leq \frac{-\xi}{T^2} \rho'(\xi/T) = \frac{d}{dt} \rho(\xi/t) \Big|_{t=T}.$$

Thus we've constructed a modulus ρ which satisfies (2.12), completing the proof of the generation of a Lipschitz modulus of continuity (1.6) in our main theorem.

9. REGULARITY IN TIME

With the construction of the modulus ρ , we get universal Lipschitz bounds in space for $f_x(t, \cdot)$. By the structure of (1.2), we also get regularity in space for f_t .

Proposition 9.1. *Let $f : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ be a classical solution to (1.2) with $\|f(t, \cdot)\|_{W^{1,\infty}}$ bounded and $\|f_{xx}(t, \cdot)\|_{L^\infty} \lesssim 1/t$. Then $f_t(t, \cdot)$ is Log-Lipschitz in space with*

$$(9.1) \quad |f_t(t, \cdot)| \lesssim \max\{-\log(t), 1\}, \quad |f_t(t, x) - f_t(t, y)| \lesssim -\log(|x-y|)|x-y| \left(1 + \frac{1}{t}\right) \quad 0 < |x-y| < 1/2.$$

Proof. For $t < 1$, we have that

$$\begin{aligned}
 |f_t(t, x)| &= \left| \int_{\mathbb{R}} \frac{\delta_h f(t, x) - h f_x(t, x)}{\delta_h f(t, x)^2 + h^2} dh \right| \leq \left| \int_0^\infty \frac{\delta_h f(t, x) + \delta_{-h} f(t, x)}{\delta_{-h} f(t, x)^2 + h^2} dh \right| \\
 (9.2) \quad &+ \left| \int_0^\infty \frac{(\delta_h f(t, x) - h f_x(t, x))(\delta_h f(t, x)^2 - \delta_{-h} f(t, x)^2)}{(\delta_h f(t, x)^2 + h^2)(\delta_{-h} f(t, x)^2 + h^2)} dh \right| \\
 &\lesssim \int_0^t \frac{1}{t} dh + \int_t^1 \frac{1}{h} dh + \int_1^\infty \frac{1}{h^2} + \frac{1}{h^3} dh \lesssim -\log(t) + 1.
 \end{aligned}$$

For $t > 1$, you can similarly show $|f_t(t, x)| \lesssim 1$, proving the first bound.

For regularity in space, we see that

$$\begin{aligned}
 (9.3) \quad f_t(t, x) - f_t(t, y) &= \int_{\mathbb{R}} \frac{\delta_h f(t, x) - h f_x(t, x)}{\delta_h f(t, x)^2 + h^2} - \frac{\delta_h f(t, y) - h f_x(t, y)}{\delta_h f(t, y)^2 + h^2} dh \\
 &= \int_{\mathbb{R}} \frac{\delta_h f(t, x) - h f_x(t, x) - (\delta_h f(t, y) - h f_x(t, y))}{\delta_h f(t, y)^2 + h^2} + \frac{(\delta_h f(t, x) - h f_x(t, x))(\delta_h f(t, x)^2 - \delta_h f(t, y)^2)}{(\delta_h f(t, x)^2 + h^2)(\delta_h f(t, y)^2 + h^2)} dh \\
 &\leq \left| \int_{|h| < |x-y|} \right| + \left| \int_{|x-y| < |h| < 1} \right| + \left| \int_{|h| > 1} \right|
 \end{aligned}$$

For $|h| < |x - y|$, we can bound similarly to before to get that

$$(9.4) \quad \left| \int_{|h| < |x-y|} \right| \lesssim \int_0^{|x-y|} \frac{1}{t} dh = \frac{|x-y|}{t}.$$

For midsize $|x - y| < |h| < 1$, we have that

$$\begin{aligned}
 (9.5) \quad \left| \delta_h f(t, x) - h f_x(t, x) - (\delta_h f(t, y) - h f_x(t, y)) \right| &= \left| \int_0^h \delta_s f_x(t, x) - \delta_s f_x(t, y) ds \right| \lesssim \frac{|x-y|h}{t}, \\
 \left| \delta_h f(t, x) - \delta_h f(t, y) \right| &= \left| \int_0^h f_x(t, x+s) - f_x(t, y+s) ds \right| \lesssim \frac{|x-y|h}{t}.
 \end{aligned}$$

Thus

$$(9.6) \quad \left| \int_{|x-y| < |h| < 1} \right| \lesssim \frac{|x-y|}{t} \int_{|x-y|}^1 \frac{1}{h} dh = \frac{-\ln(|x-y|)|x-y|}{t}.$$

Finally, we use L^∞ bounds on f to get that

$$\begin{aligned}
 (9.7) \quad \left| \int_{|h|>1} \right| &\leq \left| \int_{|h|>1} \frac{\delta_h f(t, x) - \delta_h f(t, y)}{\delta_h f(t, y)^2 + h^2} + \frac{(\delta_h f(t, x) - h f_x(t, x))(\delta_h f(t, x)^2 - \delta_h f(t, y)^2)}{(\delta_h f(t, x)^2 + h^2)(\delta_h f(t, y)^2 + h^2)} dh \right| \\
 &\quad + |f_x(t, x) - f_x(t, y)| \left| \int_{|h|>1} \frac{-h}{\delta_h f(t, y)^2 + h^2} dh \right| \\
 &\lesssim |x - y| \int_1^\infty \frac{1}{h^2} + \frac{1}{h^3} dh + \frac{|x - y|}{t} \int_1^\infty \frac{1}{h^3} dh \lesssim \left(1 + \frac{1}{t}\right) |x - y|.
 \end{aligned}$$

Putting this all together, we thus have that

$$(9.8) \quad |f_t(t, x) - f_t(t, y)| \lesssim -\ln(|x - y|) |x - y| \left(1 + \frac{1}{t}\right).$$

□

Recall that for initial data $f_0 \in W^{1,\infty}(\mathbb{R})$ with $\|f_0'\|_{L^\infty} < 1$, we approximated f_0 in $W_{loc}^{1,\infty}$ with $f_0^\epsilon \in C_c^\infty(\mathbb{R})$. The L^∞ bound on f_t proven above along with the maximum principle for f_x is enough to ensure that there a subsequence f^{ϵ_k} converging in $C([0, \infty) \times \mathbb{R})$ to a Lipschitz (weak) solution f . In order to get a classical C^1 solution, we need regularity estimates for f_x, f_t in both time and space. The modulus ρ and Proposition 9.1 give the regularity in space that we need for f_x, f_t . All that leaves is to prove regularity in time.

Proposition 9.2. *Let f be a sufficiently smooth solution to (1.2) with $\|f_x\|_{L^\infty} < 1$. Then $f_x, f_t \in C_{loc}^\alpha((0, \infty) \times \mathbb{R})$ with*

$$(9.9) \quad \|f_x\|_{C^\alpha(Q_{t/4}(t, x))}, \|f_t\|_{C^\alpha(Q_{t/4}(t, x))} \leq C(\|f_x\|_{L^\infty}, \|f\|_{L_t^\infty((t/2, 3t/2); W_x^{2,\infty}(\mathbb{R}))) \max\{t^{-\alpha}, 1\},$$

where $Q_r(s, y) = (s - r, s] \times B_r(y)$, and $\alpha > 0$ depends only on $\|f_x\|_{L^\infty}$.

Proof. We have that f_x solves

$$(9.10) \quad (f_x)_t(t, x) = f_{xx}(t, x) \int_{\mathbb{R}} \frac{-h}{\delta_h f(t, x)^2 + h^2} dh + \int_{\mathbb{R}} \delta_h f_x(t, x) K(t, x, h) dh,$$

where $\frac{\lambda}{h^2} \leq K(t, x, h) \leq \frac{\Lambda}{h^2}$ is uniformly elliptic with ellipticity constants λ, Λ depending on $\|f_x\|_{L^\infty}$. Rewriting this, we have that f_x satisfies

$$\begin{aligned}
 (9.11) \quad (f_x)_t - \int_{\mathbb{R}} \delta_h f_x(t, x) \left(\frac{K(t, x, h) + K(t, x, -h)}{2} \right) dh &= f_{xx}(t, x) \int_{\mathbb{R}} \frac{-h}{\delta_h f(t, x)^2 + h^2} dh \\
 &\quad + \int_{\mathbb{R}} \delta_h f_x(t, x) \left(\frac{K(t, x, h) - K(t, x, -h)}{2} \right) dh.
 \end{aligned}$$

Let $F(t, x)$ denote the righthand side of (9.11). Then $F(t, x)$ is locally bounded with $|F(t, x)|$ controlled by $\|f(t, \cdot)\|_{W^{2,\infty}}$. Then since $(K(t, x, h) + K(t, x, -h))/2$ is a symmetric uniformly elliptic kernel, it follows that we have local C^α bounds for $\alpha \leq \alpha_0$ for some α_0 depending on ellipticity constants (see [14]).

So, all we have to do is give bounds on $F(t, x)$ depending only on $\|f(t, \cdot)\|_{W^{2,\infty}}$. Similar to proof of Lemma 5.1,

$$(9.12) \quad \int_{\mathbb{R}} \frac{-h}{\delta_h f(t, x)^2 + h^2} dh = \int_0^\infty h \frac{\delta_h f(t, x)^2 - \delta_{-h} f(t, x)^2}{(\delta_h f(t, x)^2 + h^2)(\delta_{-h} f(t, x)^2 + h^2)} dh \lesssim \int_0^1 1 dh + \int_1^\infty \frac{1}{h^3} dh \lesssim 1.$$

Also similar to the proof of Lemma 5.2 (specifically (5.12)), we have that

$$(9.13) \quad |K(t, x, h) - K(t, x, -h)| \lesssim \min\left\{\frac{1}{h}, \frac{1}{h^3}\right\},$$

so

$$(9.14) \quad \left| \int_{\mathbb{R}} \delta_h f_x(t, x) \left(\frac{K(t, x, h) - K(t, x, -h)}{2} \right) dh \right| \lesssim \int_0^1 1 dh + \int_1^\infty \frac{1}{h^3} dh \lesssim 1.$$

Thus since we've bounded the right hand side of (9.11) depending only on $\|f(t, \cdot)\|_{W^{2,\infty}}$, we have our local C^α bounds for f_x for all α sufficiently small. A C^α bound that is uniform in x for f_x then gives a log C^α estimate for f_t , similar to the proof for regularity in space in Proposition 9.1. Thus we have C^α estimates for both f_x, f_t . \square

APPENDIX A. UNIQUENESS

We know prove that if our initial data $f_0 \in C^{1,\epsilon}(\mathbb{R})$ with $\|f'_0\|_{L^\infty} < 1$, then the solution f given by Theorem 1.1 is unique with $f \in L^\infty([0, \infty); C^{1,\epsilon})$. As mentioned before, this essentially follows from the uniqueness theorem given in [4], which under our assumptions simplifies to

Theorem A.1. (*Constantin et al*) Let $f \in L^\infty([0, T]; W^{1,\infty})$ be a classical, C^1 solution to (1.2) with initial data $f(0, x) = f_0(x)$. Assume that $\lim_{x \rightarrow \infty} f(t, x) = 0$, and that there is some modulus of continuity $\tilde{\rho}$ such that

$$(A.1) \quad f_x(t, x) - f_x(t, y) \leq \tilde{\rho}(|x - y|), \quad \forall 0 \leq t \leq T, \quad x \neq y \in \mathbb{R}.$$

Then the solution f is unique.

The authors of [4] note that the uniform continuity assumption should be the only real assumption; the decay is assumed for convenience in their proof. So, we start by proving that if $f_0 \in C^{1,\epsilon}(\mathbb{R})$, then the solution $f \in L^\infty([0, \infty); C^{1,\epsilon})$. To begin, suppose that $f_0 \in C^{1,1}(\mathbb{R})$. Then necessarily f'_0 has modulus $\rho(\cdot/\delta)$ for some $\delta > 0$ sufficiently small. The same proof for the instantaneous generation of the modulus ρ will give that $f_x(t, \cdot)$ has modulus $\rho(\cdot/t + \delta)$. Hence $f_x(t, \cdot)$ has modulus $\rho(\cdot/\delta)$ for all $t \geq 0$.

If $f_0 \in C^{1,\epsilon}(\mathbb{R})$, we can make the same essential argument by changing the definition of ρ, ω . You can repeat the arguments of section 7 and 8 for the modulus

$$(A.2) \quad \begin{cases} \omega^{(\epsilon)}(\xi) = \xi^\epsilon, & 0 \leq \xi \leq \delta \\ \omega^{(\epsilon)'}(\xi) = \frac{\gamma}{\xi(4 + \log(\xi/\delta))}, & \xi \geq \delta \end{cases}.$$

All the error terms for $\xi \leq \delta$ are of order $\xi^{2\epsilon-1}$, while the diffusion term is of the order $\xi^{\epsilon-1}$, so there are now problems as long as δ is sufficiently small. The argument for $\xi \geq \delta$ is identical to the original. Taking $\rho^{(\epsilon)}$ to be some suitable rescaling of $\omega^{(\epsilon)}$, we then have that if f'_0 has modulus $\rho^{(\epsilon)}(\cdot/\delta)$, then $f_x(t, \cdot)$ will have modulus $\rho^{(\epsilon)}(\cdot/t + \delta)$.

Thus if $f_0 \in C^{1,\epsilon}(\mathbb{R})$, then the solution f given by Theorem 1.1 will satisfy the main uniform continuity assumption of Theorem A.1. Our solution f will not decay as $x \rightarrow \infty$, but that assumption isn't truly necessary.

Let f_1, f_2 be two uniformly continuous, classical solutions to (1.2) with the same initial data, and let $M(t) = \|f_1(t, \cdot) - f_2(t, \cdot)\|_{L^\infty}$. With the decay assumption, the authors of [4] are able to assume that for almost every t , there is a point $x(t) \in \mathbb{R}$ such that

$$(A.3) \quad M(t) = |f_1(t, x(t)) - f_2(t, x(t))|, \quad \frac{d}{dt}M(t) = \left(\frac{d}{dt}|f_1 - f_2|\right)(t, x(t)).$$

They then bound $\frac{d}{dt}|f_1(t, x(t)) - f_2(t, x(t))|$ using equation (1.2), $\tilde{\rho}$, and $W^{1,\infty}$ bounds.

Without the decay assumption, you instead use that

$$(A.4) \quad \frac{d}{dt}M(t) \leq \sup\left\{\frac{d}{dt}|f_1(t, x) - f_2(t, x)| : |f_1(t, x) - f_2(t, x)| \geq M(t) - \delta\right\},$$

where $\delta > 0$ is arbitrary. When you go to bound $\frac{d}{dt}|f_1(t, x) - f_2(t, x)|$, you then get new error terms which can be bounded by

$$(A.5) \quad C(\tilde{\rho}, \max_i \|f_i(t, \cdot)\|_{W^{1,\infty}}, M(t)) (\delta + |f_{1,x}(t, x) - f_{2,x}(t, x)|).$$

Since $f_{i,x}(t, x)$ is bounded and has modulus $\tilde{\rho}$, it then follows that

$$(A.6) \quad |f_{1,x}(t, x) - f_{2,x}(t, x)| = o(\delta).$$

Thus by taking δ sufficiently small depending on $\tilde{\rho}, \max_i \|f_i(t, \cdot)\|_{W^{1,\infty}}, M(t)$, we can guarantee that the new error terms $\lesssim M(t)$. Then the original proof of [4] goes through.

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